University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2011
Mathematics 1201
Friday 14 January 2011 1.30 – 3.30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

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1) (i) Negate the formula $(q \Rightarrow \neg p) \wedge (p \Rightarrow \neg r)$ and replace the resulting expression by an equivalent one which does not involve $\neg$ or $\Rightarrow$.

(ii) Replace the following expression

$$((\exists x)(\forall y)P(x,y) \wedge (\forall x)(\exists y)\neg Q(x,y)) \bigvee \neg(\forall x)(\forall y)\neg R(x,y)$$

by an equivalent one which does not involve $\neg$, $\vee$ or $\wedge$.

Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that (a) f is injective ; (b) f is surjective.

Let $\mathcal{I} = \{y \in \mathbf{R} \mid -1 < y < 1\}$; in (iii) and (iv) below decide, justifying your statements, whether or not the given mapping is

(a) injective (b) surjective :

(iii) $g : \mathbf{Z} \to \mathcal{I}$; $g(x) = \dfrac{x}{x^2+1}$; (iv) $h : \mathbf{R} \to \mathcal{I}$; $h(x) = \dfrac{x}{x^2+1}$.

PLEASE TURN OVER

2) i) Explain what is meant by saying that a mapping $f : A \to B$ is *invertible*. Prove that f is invertible if and only if f is bijective.

ii) Decompose the following permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 11 & 5 & 7 & 13 & 12 & 10 & 2 & 9 & 1 & 8 & 6 & 3 & 4 \end{pmatrix}$$

into a product of disjoint cycles and hence compute $\text{ord}(\sigma)$ and $\text{sign}(\sigma)$.

iii) For the matrix A below, find A^{-1} and express A^{-1} as a product of elementary matrices; hence also express A as a product of elementary matrices.

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

3) Let V, W be vector spaces over a field $\mathbf{F}$ and let $T : V \to W$ be a linear mapping; explain what is meant by

(a) the kernel, $\text{Ker}(T)$ and (b) the image, $\text{Im}(T)$.

State without proof a relationship which holds between $\dim \text{Ker}(T)$ and $\dim \text{Im}(T)$.

Find the general solution to the system $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 3 & -1 & -1 & -3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 5 \end{pmatrix}.$$

Denoting by $T_A : \mathbf{Q}^7 \to \mathbf{Q}^4$ the linear map $T_A(\mathbf{x}) = A\mathbf{x}$, find also (i) a basis for $\text{Ker}(T_A)$ and (ii) a basis for $\text{Im}(T_A)$.

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4) Let $T : U \to V$ be a linear map between vector spaces U, V, and let $\mathcal{E} = (e_i)_{1 \le i \le m}$ be a basis for U and $\Phi = (\varphi_j)_{1 \le j \le n}$ be a basis for V.

Explain what is meant by the matrix $\mathcal{M}(T)^{\Phi}_{\mathcal{E}}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right).

If $S : V \to W$ is also linear and $\Psi = (\psi_k)_{1 \le k \le p}$ is a basis for W prove that

$$\mathcal{M}(S \circ T)^{\Psi}_{\mathcal{E}} = \mathcal{M}(S)^{\Psi}_{\Phi} \mathcal{M}(T)^{\Phi}_{\mathcal{E}}.$$

Let $T : \mathbf{Q}^3 \to \mathbf{Q}^3$ be the mapping $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 & & \\ 2x_1 & +x_2 & +2x_3 \\ x_1 & & +3x_3 \end{pmatrix}$

and let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\Phi = \left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Write down (i) $\mathcal{M}(T)^{\mathcal{E}}_{\mathcal{E}}$ and (ii) $\mathcal{M}(\mathrm{Id})^{\mathcal{E}}_{\Phi}$. Hence find $\mathcal{M}(T)^{\Phi}_{\Phi}$.

5) Let V be the vector space consisting of all functions $f : \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) = \lambda_1 \sin(2x) + \lambda_2 \cos(2x) + \lambda_3 x\sin(2x) + \lambda_4 x\cos(2x) \qquad (\lambda_i \in \mathbf{Q})$$

and let $D : V \to V$ be the linear map $D(f) = \frac{df}{dx}$. Taking

$$\{ \sin(2x),\ \cos(2x),\ x\sin(2x),\ x\cos(2x) \}$$

as basis for V find :

i) the matrix of D ; ii) the matrix of D^3 ; iii) the matrix of D^{-1}.

Hence *without further explicit differentiation or integration* write down

iv) $\dfrac{d^3}{dx^3}(\cos(2x) + x\sin(2x))$

v) $\displaystyle\int \{ \sin(2x) + x\sin(2x) + x\cos(2x) \} \, dx$

[You may ignore the constant of integration in v).]

PLEASE TURN OVER

6) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V. Explain what is meant by saying that

i) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is *linearly independent* ;

ii) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *spans* V.

Suppose that the subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V and that $\mathbf{v}_n$ can be expressed as a linear combination

$$\mathbf{v}_n = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_{n-1} \mathbf{v}_{n-1}.$$

Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ also spans V.

Let $\mathbf{F}$ be a field and consider the following vectors in $\mathbf{F}^4$;

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Decide with proof whether the above vectors are linearly independent

a) when $\mathbf{F} = \mathbf{Q}$;

b) when $\mathbf{F} = \mathbf{F}_2$ is the field with two elements.

Should you decide that the vectors are *not* linearly independent then give an explicit dependence relation between them.

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